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Calculation of the Addition Coefficients in Electromagnetic Multisphere-Scattering Theory

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One of the most intractable problems in electromagnetic multisphere-scattering theory is the formulation and evaluation of vector addition coefficients introduced by the addition theorems for vector spherical harmonics. This paper presents an efficient approach for the calculation of both scalar and vector translational addition coefficients, which is based on fast evaluation of the Gaunt coefficients. The paper also rederives the analytical expressions for the vector translational addition coefficients and discusses the strengths and limitations of other formulations and numerical techniques found in the literature. Numerical results from the formulation derived in this paper agree with those of a previously published recursion scheme that completely avoids the use of the Gaunt coefficients, but the method of direct calculation proposed here reduces the computing time by a factor of 4–6.

I. INTRODUCTION

Electromagnetic scattering characteristics of compact multisphere systems are of considerable interest for a wide range of applications, from modeling the interplanetary and interstellar dust clouds to enhancing the gain along preferred directions of an antenna. Not surprisingly, this has stimulated active research in the theoretical study of electromagnetic scattering by arbitrary configurations of neighboring spheres in the last few decades [1–26]. The formulation of the interactive scattering by an ensemble of spheres is complicated, both analytically and numerically, as it demands the use of vector addition theorems. These addition theorems or expansions are necessary to describe an electromagnetic field using the same set of basis spherical wave functions, with respect to a reference system with an arbitrary origin. Major difficulty encountered in the application of the addition theorems is the evaluation of the addition coefficients. The derivation of the analytical expressions for the addition coefficients is lengthy and complex and it is difficult to ensure that the final formulae are error-free. Even if the problem is suitably formulated, the computation of the addition coefficients is annoyingly time-consuming even with the reasonably fast computers available nowadays. This necessitates the use of efficient algorithms.

Rotational addition theorems are well known for scalar spherical wave functions [27–30] and well formulated for vector spherical harmonics [25, 26]. Therefore, this paper concentrates on the calculation of the scalar and vector translational addition coefficients. Common to the analytical expressions for these coefficients is a product of two associated Legendre functions that in turn may be expressed in terms of a linearization expansion involving the Gaunt coefficients [31]. Considerable research work has been conducted on the calculation of the Gaunt coefficients, including those of Bruning [32], Fuller [33] and Xu [34]. Stein [25] and, later, Mackowski [21] showed that the vector translational addition coefficients can be evaluated from pertinent scalar addition coefficients. Correct recurrence relations for the scalar addition coefficients were obtained by Mackowski [21] and were consequently used to circumvent the use of the Gaunt coefficient in the calculation of the vector translational addition coefficients at the cost of evaluating certain auxiliary quantities. Cruzan [26] also derived expressions for the vector translational addition coefficients using the Wigner 3*jm* symbols [35, 36]. The expressions of Cruzan's type are more efficient in practical applications. However, Cruzan's expression for one of the two classes of vector translational addition coefficients is incorrect and needs a replacement.

In this paper we have solved most of the problems in the formulation and evaluation of the vector translational addition coefficients for general multisphere scattering calculations. Since the Gaunt coefficients are inseparably connected with the translational addition coefficients, the methods for calculating the Gaunt coefficients are addressed first in Section II. Both our linearization method and the recurrence formulae developed by Bruning and by Fuller are discussed. In Section III, we then proceed to elucidate the new approach to directly calculating the scalar translational addition coefficients and we also discuss Mackowski's recursive approach. Section IV reviews three sets of expressions for the vector translational addition coefficients formulated by Stein, Cruzan, and Mackowski,

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respectively, and it also presents our revised version of Cruzan's expressions, which enable us to greatly reduce the CPU time requirement in practical computation. Discussions of the numerical methods are presented in Section V along with comparisons of the efficiency and accuracy of the results to those of other methods. Some practical applications of this formulation are given in Section VI. Finally, Section VII concludes with a brief summary of the method and closing remarks.

II. GAUNT COEFFICIENTS

The Gaunt coefficient is defined by

$$a(m, n, \mu, \nu, p) = \frac{(2p+1)}{2} \frac{(p-m-\mu)!}{(p+m+\mu)!} \int_{-1}^{1} P_n^m(x) P_\nu^\mu(x) P_p^{m+\mu}(x)\, dx, \tag{1}$$

where m, n, μ, ν, p are integers and $n \geq 0$, $\nu \geq 0$, $p \geq 0$. This integral of the product of three associated Legendre functions was first used by Gaunt in 1929 [31] in his study of the atomic structure of helium triplets. The Gaunt coefficients vanish whenever $p = n + \nu - 1, n + \nu - 3, \ldots$. Xu [34] formulated an alternative equivalent definition,

$$P_n^m(x) P_\nu^\mu(x) = \sum_{q=0}^{q_{\max}} a_q P_{n+\nu-2q}^{m+\mu}(x), \tag{2}$$

where a_q is an abbreviated notation for the Gaunt coefficient $a(m, n, \mu, \nu, n + \nu - 2q)$, and

$$q_{\max} = \min\left(n, \nu, \frac{n + \nu - |m + \mu|}{2}\right). \tag{3}$$

This definition of the Gaunt coefficients is particularly convenient for the multisphere scattering problem wherein the Gaunt coefficients are usually computed as a group.

Gaunt coefficients are closely related to the Clebsch–Gordan coefficients that are extensively used in quantum mechanics [37, 38] and usually expressed in terms of the Wigner $3jm$ symbols. Cruzan [26] expressed the Gaunt coefficients (and, therefore, the vector addition coefficients) in terms of the Wigner $3jm$ symbols as well. The integral over the triple associated Legendre functions and, thus, the Gaunt coefficient has been evaluated in various forms. It was presented as complex summations of factorials by Gaunt and others and was identified as a generalized hypergeometric series by Hardy [39] and by Rose [40]. With Cruzan's representation, the Gaunt coefficient is given by

$$a(m, n, \mu, \nu, p) = (-1)^{m+\mu}(2p+1) \times \left[\frac{(n+m)!(\nu+\mu)!(p-m-\mu)!}{(n-m)!(\nu-\mu)!(p+m+\mu)!}\right]^{1/2} \times \begin{pmatrix} n & \nu & p \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n & \nu & p \\ m & \mu & -m-\mu \end{pmatrix}, \tag{4}$$

where $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ is the Wigner $3jm$ symbol. This $3jm$ formulation may be considered elegant, but it works well only for low degrees (small values of n and ν) [34], while multisphere scattering problems usually involve both low and high degrees. Also, it is too time consuming to be of practical use in actual multisphere scattering calculations.

Recently, Xu [34] has devised an efficient way to calculate the Gaunt coefficients that works satisfactorily for both low and high degrees and requires only around 1% of the computing time demanded by the $3jm$ formulation. This algorithm defines the "normalized Gaunt coefficient" by $\tilde{a}_k = a_k/a_0$. By definition, $\tilde{a}_0 \equiv 1$. The first Gaunt coefficient a_0 is obtained directly from

$$a_0 = \frac{(n+1)_n(\nu+1)_\nu}{(n+\nu+1)_{n+\nu}} \frac{(n+\nu-m-\mu)!}{(n-m)!(\nu-\mu)!}. \tag{5}$$

In general, all other $q_{\max}$ nonzero normalized Gaunt coefficients are given by

$$\tilde{a}_q = \frac{(p+1/2)_{2q}}{(-n_4)_{2q}} \sum_{k=0}^{q} \frac{(m-n)_{2k}(\mu-\nu)_{2q-2k}}{k!(q-k)!(-n+1/2)_k(-\nu+1/2)_{q-k}} - \sum_{j=0}^{q-1} \frac{(-p-q+j+1/2)_{q-j}}{(q-j)!} \tilde{a}_j, \tag{6}$$

where $p = n + \nu - 2q$, $n_4 = n + \nu - m - \mu$, and $q = 1, 2, \ldots, q_{\max}$. From this algorithm, the second normalized Gaunt coefficient can be expressed in the form

$$\tilde{a}_1 = \frac{(2n+2\nu-3)}{2}\left\{1 - \frac{(2n+2\nu-1)}{n_4(n_4-1)} \times \left[\frac{(m-n)(m-n+1)}{(2n-1)} + \frac{(\mu-\nu)(\mu-\nu+1)}{(2\nu-1)}\right]\right\} \tag{7}$$

and others can be expressed in a similar way but with more complicated forms. The lower triangular system of linear equations (6) is also easy to solve numerically. All coefficients and constant terms in the linear equations have simple two-term recurrence relations with all starting values being 1 [34].

It is noteworthy here that Bruning and Lo [8, 32], as well as Fuller [33], have derived some general three-term recurrence relations for the Gaunt coefficients. A recursive approach to evaluating these coefficients is highly desirable in practical applications. In 1971, Bruning and Lo [8] published the three-term recurrence formula for the particular Gaunt coefficients with $\mu = -m$,

$$\alpha_{p-3}a_{p-4} + (4m^2 - \alpha_{p-2} - \alpha_{p-1})a_{p-2} + \alpha_p a_p = 0, \quad (8)$$

where a_p stands for the Gaunt coefficient $a(m, n, -m, \nu, p)$ and

$$\alpha_p = \frac{[p^2 - (n+\nu+1)^2][p^2 - (n-\nu)^2]}{1 - 4p^2}. \quad (9)$$

Explicit expressions, given by Bruning, for the two starting values for the implementation of the above recurrence formula are essentially the same as Eqs. (5) and (7). Equation (8) provides very accurate numerical results for all low- and high-degree coefficients, but this recurrence formula applies only to the special case where coordinate translation is restricted to be along the z-axis. By the use of six recurrence relations listed by Cruzan [26] involving as many as six terms or three indices within a single formula, Bruning [32] also obtained the following pair of three-term recurrence formulas in two indices ν and p for the case of a general coordinate translation:

$$\begin{aligned}&(2p+1)(2p-3)(\nu-\mu+1)[p(p-1)(m-\mu)\\&\quad -(m+\mu)(n-\nu)(n+\nu+1)]\\&\quad \times a(m,n,\mu,\nu+1,p-1)\\&\quad +(2p+1)(p-m-\mu-1)(p-n+\nu-1)\\&\quad \times(p+n+\nu)[\mu(p-\nu-1)-m(\nu+1)]\\&\quad \times a(m,n,\mu,\nu,p-2)\\&\quad +(2p-3)(p+m+\mu)(n+\nu-p+1)\\&\quad \times(n-\nu+p)[\mu(p+\nu)+m(\nu+1)]\\&\quad \times a(m,n,\mu,\nu,p)=0, \qquad (10)\end{aligned}$$

$$\begin{aligned}&(2p+1)(2p-3)(\nu+\mu)[p(p-1)(m-\mu)\\&\quad -(m+\mu)(n-\nu)(n+\nu+1)]\\&\quad \times a(m,n,\mu,\nu-1,p-1)\\&\quad +(2p+1)(p-m-\mu-1)(n+\nu-p+2)\\&\quad \times(n-\nu+p-1)[\mu(p+\nu)+m\nu]\\&\quad \times a(m,n,\mu,\nu,p-2)\\&\quad +(2p-3)(p+m+\mu)(n+\nu+p+1)\\&\quad \times(p-n+\nu)[\mu(p-\nu-1)-m\nu]\\&\quad \times a(m,n,\mu,\nu,p)=0. \qquad (11)\end{aligned}$$

As a further development of Bruning's results, Fuller [33] derived a single three-term recurrence relation in the index p alone:

$$\chi_1 a_{p-4} + \chi_2 a_{p-2} + \chi_3 a_p = 0, \quad (12)$$

where a_p denotes the Gaunt coefficient $a(m, n, \mu, \nu, p)$ and

$$\begin{aligned}\chi_1 &= A(\nu+1,p-1)B(-\nu-2,p-3)B(\nu,p-4),\\ \chi_2 &= [(\nu+1)^2-\mu^2]A(\nu+1,p-1)A(\nu+1,p-3)\\&\quad \times A(\nu+2,p-2) \qquad (13)\\&\quad + A(\nu+1,p-3)B(\nu,p-2)\Gamma(-\nu-2,p-1)\\&\quad + A(\nu+1,p-1)B(-\nu-2,p-3)\Gamma(\nu,p-2),\\ \chi_3 &= A(\nu+1,p-3)\Gamma(-\nu-2,p-1)\Gamma(\nu,p),\end{aligned}$$

with

$$\begin{aligned}A(\nu,p) &= p(p+1)(m-\mu)-(m+\mu)(n-\nu+1)(n+\nu),\\ B(\nu,p) &= \frac{p-m-\mu+1}{2p+1}(n-\nu-p-1)(n+\nu+p+2)\\&\quad \times[m(\nu+1)-\mu(p-\nu+1)], \qquad (14)\\ \Gamma(\nu,p) &= \frac{p+m+\mu}{2p+1}(n+\nu-p+1)(n-\nu+p)\\&\quad \times[m(\nu+1)+\mu(p+\nu)].\end{aligned}$$

This recursion scheme works well when the degree involved is not too high. As pointed out by Fuller [33], the practical application of Eq. (12) is, however, complicated by the fact that χ_1, the coefficient of the $p - 4$ term, vanishes for certain relations between m, n, μ, ν, and p. Fuller [33] has investigated all the seven classes of zeros of χ_1. Most of these cases can be handled without difficulty. But in some cases, $\chi_1 = 0$ cannot be simply avoided. For certain integer groups (m, n, μ, ν) with some particular values of p, such as (−2, 7, 3, 8) or (3, 8, −2, 7) with $p = 5$, $B(-\nu-2, p-3) = 0$ while transposing the indices (m, n) with (μ, ν) results in $B(\nu, p-4) = 0$. In these cases the recurrence relation represented by Eq. (12) is not applicable and other methods for calculating these particular coefficients must be employed. Also, for some groups of (m, n, μ, ν) with very high degree n or ν, such as (1, 40, 45, 46) and (1, 30, 40, 60), Eq. (12) is somehow unable to produce accurate numerical results. Further research work may erase these imperfections and consummate the existing general recursion scheme.

III. SCALAR TRANSLATIONAL ADDITION COEFFICIENTS

In 1954, Friedman and Russek [24] derived a general translational addition theorem for the spherical scalar wave functions. Stein [25] later revised it and extended it to the vector case; see also Cruzan [26]. Since the vector addition theorems are derived from the scalar addition theorems and vector addition coefficients can be expressed in terms of scalar addition coefficients, we treat the calculation of the scalar addition coefficients first.

A. *Scalar Translational Addition Theorem*

The potentials or scalar spherical waves are solutions of the Helmholtz equation in spherical coordinates,

$$\nabla^2 u + k^2 u = 0, \tag{15}$$

where $k^2 = \omega^2 \varepsilon\mu$ (k is the wave number), ω is the circular frequency, and ε, μ are the permeability and the dielectric constant of the medium, respectively. The set of characteristic solutions with suppressed time dependence $\exp(-i\omega t)$ is given by

$$u_{mn}^{(J)} = z_n^{(J)}(kr) P_n^m(\cos\theta)\exp(im\phi), \tag{16}$$

where $z_n^{(J)}$ is appropriately selected among the following four spherical functions: (a) the Bessel function of the first kind j_n, (b) the Bessel function of the second kind y_n, (c) the Hankel function of the first kind $h_n^{(1)}$, and (d) the Hankel function of the second kind $h_n^{(2)}$, denoted by $J = 1, 2, 3$, or 4, respectively; $P_n^m(\cos\theta)$ is the associated Legendre function of the first kind and of degree n and order m, n and m are integers with $0 \le n < \infty$ and $|m| \le n$.

The translation of an elementary spherical wave $u_{\mu\nu}$ from the lth coordinate system to the jth coordinate system is given by [26]

$$\begin{aligned} u_{\mu\nu}^{(J)}(r_l, \theta_l, \phi_l) &= \sum_{n=1}^{\infty}\sum_{m=-n}^{n} C_{mn}^{\mu\nu}(d_{lj}, \theta_{lj}, \phi_{lj}) u_{mn}^{(1)}(r_j, \theta_j, \phi_j), \\ &\qquad r \le d_{lj}, \\ u_{\mu\nu}^{(J)}(r_l, \theta_l, \phi_l) &= \sum_{n=1}^{\infty}\sum_{m=-n}^{n} C_{mn}^{\mu\nu}(d_{lj}, \theta_{lj}, \phi_{lj}) u_{mn}^{(J)}(r_j, \theta_j, \phi_j), \\ &\qquad r \ge d_{lj}, \end{aligned} \tag{17}$$

where the scalar addition coefficients are given by Stein [25] as

$$\begin{aligned} C_{mn}^{\mu\nu} &= (-1)^m (2n+1) i^{n-\nu} \sum_p i^p a(-m, n, \mu, \nu, p) \\ &\quad \times z_p^{(J)}(kd_{lj}) P_p^{\mu-m}(\cos\theta_{lj}) \exp[i(\mu - m)\phi_{lj}], \quad r \le d_{lj}, \\ C_{mn}^{\mu\nu} &= (-1)^m i^{n-\nu} \sum_p i^p (2p+1) a(m-\mu, n, \mu, \nu, p) \\ &\quad \times j_p(kd_{lj}) P_p^{\mu-m}(\cos\theta_{lj}) \exp[i(\mu - m)\phi_{lj}], \quad r \ge d_{lj}. \end{aligned} \tag{18}$$

Here, $(d_{lj}, \theta_{lj}, \phi_{lj})$ are the spherical coordinates of the origin of the jth coordinate system in the lth coordinate system and $z_p^{(J)}$ is usually taken as $h_p^{(1)}$. By using the identity relation

$$a(m-\mu, n, \mu, \nu, p) \equiv (-1)^\mu \frac{2n+1}{2p+1} a(-m, n, \mu, \nu, p), \tag{19}$$

Bruning [32] obtained the more symmetric form

$$C_{mn}^{\mu\nu} = (-1)^m (2n+1) i^{n-\nu} \sum_p i^p a(-m, n, \mu, \nu, p) \times \begin{pmatrix} z_p^{(J)}(kd_{lj}) \\ j_p(kd_{lj}) \end{pmatrix} P_p^{\mu-m}(\cos\theta_{lj}) \exp[i(\mu - m)\phi_{lj}] \quad \begin{pmatrix} r \le d_{lj} \\ r \ge d_{lj} \end{pmatrix}. \tag{20}$$

With the use of the spherical coordinates of the origin of the lth coordinate system in the jth coordinate system $(d_{jl}, \theta_{jl}, \phi_{jl})$ and the relations

$$d_{jl} = d_{lj}, \quad \theta_{jl} = \pi - \theta_{lj}, \quad \phi_{jl} = \pi + \phi_{lj}, \tag{21}$$

Eq. (20) can alternatively be written in the form

$$C_{mn}^{\mu\nu} = (-1)^m (2n+1) i^{n-\nu} \sum_p (-i)^p a(-m, n, \mu, \nu, p) \times \begin{pmatrix} z_p^{(J)}(kd_{jl}) \\ j_p(kd_{jl}) \end{pmatrix} P_p^{\mu-m}(\cos\theta_{jl}) \exp[i(\mu - m)\phi_{jl}] \quad \begin{pmatrix} r \le d_{jl} \\ r \ge d_{jl} \end{pmatrix}. \tag{22}$$

B. *Calculation of the Scalar Translational Addition Coefficients*

There are two ways to calculate scalar translational addition coefficients. As shown in the preceding subsection, these coefficients can be expressed in terms of the Gaunt coefficients together with the Bessel and the associated Legendre functions. We do not need to discuss the calculation of the Bessel and the associated Legendre functions since standard methods for their evaluation are available. Thus, by the use of an efficient algorithm for the evaluation of the Gaunt coefficients (such as the methods described in Section II), it is feasible in practice to calculate the scalar translational addition coefficients directly and expeditiously. This shall henceforth be referred to as the method of direct calculation. An alternative is the recursive approach devised by Mackowski. Both of them are discussed below.

1. THE METHOD OF DIRECT CALCULATION. Any scalar addition coefficient, $C_{mn}^{\mu\nu}$, can be computed by the following two steps:

(a) All $q_{\max} + 1$ nonzero normalized Gaunt coefficients $\tilde{a}_q = \tilde{a}(-m, n, \mu, \nu, n + \nu - 2q)$ can be accurately determined by using the efficient algorithms described in Section II. Here, $q_{\max}$ takes the smallest value among n, ν, and $(n + \nu - |\mu - m|)/2$.

(b) The value of $C_{mn}^{\mu\nu}$ can then be easily evaluated by

$$C_{mn}^{\mu\nu} = (-1)^{m+n}(2n + 1)a_0 \exp[i(\mu - m)\phi_{lj}] \times \sum_{q=0}^{q_{\max}} (-1)^q \tilde{a}_q \begin{pmatrix} z_{n+\nu-2q}^{(J)}(kd_{lj}) \\ j_{n+\nu-2q}(kd_{lj}) \end{pmatrix} P_{n+\nu-2q}^{\mu-m}(\cos\theta_{lj}) \begin{pmatrix} r \le d_{lj} \\ r \ge d_{lj} \end{pmatrix}, \tag{23}$$

where $a_0 = a(-m, n, \mu, \nu, n + \nu)$ can be directly evaluated from Eq. (5).

2. MACKOWSKI'S RECURSION SCHEME. The basic three recurrence relations for scalar translational addition coefficients obtained by Mackowski [21] are:

$$\frac{C_{mn}^{\mu\nu-1} + C_{mn}^{\mu\nu+1}}{2\nu + 1} = \frac{C_{m-1n-1}^{\mu-1\nu}}{2n - 1} + \frac{C_{m-1n+1}^{\mu-1\nu}}{2n + 3}, \tag{24}$$

$$\frac{(\nu + \mu)(\nu + \mu + 1)C_{mn}^{\mu\nu-1} + (\nu - \mu)(\nu - \mu + 1)C_{mn}^{\mu\nu+1}}{2\nu + 1} = \frac{(n - m)(n - m - 1)}{2n - 1} C_{m+1n-1}^{\mu+1\nu} + \frac{(n + m + 1)(n + m + 3)}{2n + 3} C_{m+1n+1}^{\mu+1\nu}, \tag{25}$$

$$\frac{(\nu + \mu)C_{mn}^{\mu\nu-1} - (\nu - \mu + 1)C_{mn}^{\mu\nu+1}}{2\nu + 1} = -\frac{n - m}{2n - 1} C_{mn-1}^{\mu\nu} + \frac{n + m + 1}{2n + 3} C_{mn+1}^{\mu\nu}. \tag{26}$$

There is a special case in which the scalar addition coefficients have very simple forms and the calculation of the Gaunt coefficient is not actually involved. When $\mu = \nu = 0$,

$$C_{mn}^{00} = (-1)^{m+n}(2n + 1) \begin{pmatrix} z_n^{(J)}(kd_{lj}) \\ j_n(kd_{lj}) \end{pmatrix} P_n^{-m}(\cos\theta_{lj}) \exp(-im\phi_{lj}) \begin{pmatrix} r \le d_{lj} \\ r \ge d_{lj} \end{pmatrix}. \tag{27}$$

From Eqs. (3) and (23) it is evident that only one Gaunt coefficient, namely a_0, is involved in this case and that $a_0 = \tilde{a}_0 = 1$. The value of C_{mn}^{00} for any (m, n) can thus be directly calculated from Eq. (27) without carrying out any real calculation of the Gaunt coefficient. With these quantities as starting values, the three recurrence relations (24)–(26) allow one to determine the value of any $C_{mn}^{\mu\nu}$. The procedure for calculating a particular value of $C_{mn}^{\mu\nu}$ is as follows.

(a) Calculate the $\nu + 1$ starting values of $C_{m-\mu n+\nu-2j}^{00}$ ($j = 0, 1, \ldots, \nu$) by the use of Eq. (27).
There are, however, only $k + 1$ nonzero values, where

$$k = \min\left(\nu, \frac{n + \nu - |m - \mu|}{2}\right). \tag{28}$$

For any $j > k$, $C_{m-\mu n+\nu-2j}^{00} = 0$.

(b) Calculate the $\nu - |\mu| + 1$ values of $C_{ms}^{\mu|\mu|}$, where

$$s = n + \nu - |\mu| - 2j, \quad j = 0, 1, \ldots, \nu - |\mu|. \tag{29}$$

Only $l + 1$ of these scalar coefficients have nonzero values and

$$l = \min\left(\nu - |\mu|, \frac{n + \nu - |\mu| - |m|}{2}\right). \tag{30}$$

The calculation uses one of the two recurrence relations,

$$C_{ws}^{t+1t+1} = (2t + 1)\left(\frac{C_{w-1s-1}^{tt}}{2s - 1} + \frac{C_{w-1s+1}^{tt}}{2s + 3}\right), \tag{31}$$

$$C_{ws}^{-t-1t+1} = \frac{1}{2(t + 1)}\left[\frac{(s - w)(s - w - 1)}{2s - 1} C_{w+1s-1}^{-tt} + \frac{(s + w + 1)(s + w + 2)}{2s + 3} C_{w+1s+1}^{-tt}\right], \tag{32}$$

which are Eqs. (24) and (25) in the special case $\nu = |\mu|$. If μ is positive, the use of Eq. (31) is required and, otherwise, Eq. (32) should be used. First, ν values of the next degree of $C_{m-\mu+1n+\nu-1-2j}^{11}$ ($\mu > 0$) or $C_{m-\mu-1n+\nu-1-2j}^{-11}$ ($\mu < 0$) ($j = 0, 1, \ldots, \nu - 1$) are calculated from the $\nu + 1$ starting values obtained in the previous step (a) by using the appropriate recurrence relation (31) or (32). The same process is then repeated $|\mu|$ times until the required $\nu - |\mu| + 1$ values of $C_{ms}^{\mu|\mu|}$ are obtained. Each time, the number of scalar addition coefficients calculated is decreased by one, but in the first $\nu - k$ steps the computation needs to be carried out only for $k + 1$ quantities.

(c) Calculate $C_{mn}^{\mu\nu}$. With the use of the $\nu - |\mu| + 1$ values of $C_{ms}^{\mu|\mu|}$ obtained in step (b) and the condition

$$C_{ms}^{\mu|\mu|-1} = 0, \tag{33}$$

$\nu - |\mu|$ values of $C_{mn+\nu-|\mu|-1-2j}^{\mu|\mu|+1}$ ($j = 0, 1, \ldots, \nu - |\mu| - 1$) are found from the recurrence relation (26). Similar to the process in step (b), the procedure is repeated $\nu - |\mu|$ times until $C_{mn}^{\mu\nu}$ is reached and each time the number of scalar addition coefficients calculated is decreased by one.

(d) In order to save computing time, the relation

$$C_{mn}^{\mu\nu} = (-1)^{m+n+\mu+\nu}(2n+1)(2\nu+1)C_{-\mu\nu}^{-mn} \tag{34}$$

can be used when $\nu > n$.

Although this approach does not involve the Gaunt coefficients, a certain number of auxiliary quantities must be calculated.

We tested the procedures suggested above for both the direct and recursive methods numerically. For all the calculated integer combinations of (m, n, μ, ν), from degree $n = \nu = 1$ to $n = \nu = 30$, the numerical values of any $C_{mn}^{\mu\nu}$ obtained by either method are identical (within the accuracy allowed by the computer). The direct method is faster, however; for instance, the direct method needs 2539 CPU seconds while the recursive approach requires 3956 CPU seconds for the computation of 921,600 scalar translational addition coefficients for all the possible combinations of (m, n, μ, ν) in a range of degrees from 1 to 30 on an IBM RS6000-340 workstation.

IV. VECTOR TRANSLATIONAL ADDITION COEFFICIENTS

Addition theorems for the vector spherical wave functions are the extension of the addition theorems for scalar spherical wave functions in the vector case. This section discusses the analytical expressions for the vector translational addition coefficients.

A. *Vector Addition Theorems and the Vector Translational Coefficients*

Time-harmonic electric and magnetic fields **E** and **H** in a sourceless, isotropic, and homogeneous medium are divergence-free and must satisfy the vector wave equations

$$\nabla \times \nabla \times \mathbf{E} - k^2\mathbf{E} = 0, \quad \nabla \times \nabla \times \mathbf{H} - k^2\mathbf{H} = 0. \tag{35}$$

The linear independent vector field solutions of the vector wave equations are the vector spherical functions **M** and **N** that, in spherical polar coordinates (r, θ, ϕ), may be written in the specific component form,

$$\begin{aligned}\mathbf{M}_{mn}^{(J)} &= [\mathbf{i}_\theta i\pi_{mn}(\cos\theta) - \mathbf{i}_\phi\tau_{mn}(\cos\theta)]z_n^{(J)}(kr)\exp(im\phi),\\ \mathbf{N}_{mn}^{(J)} &= \mathbf{i}_\mathbf{r} n(n+1)P_n^m(\cos\theta)\frac{z_n^{(J)}(kr)}{kr}\exp(im\phi)\\ &\quad + [\mathbf{i}_\theta\tau_{mn}(\cos\theta) + \mathbf{i}_\phi i\pi_{mn}(\cos\theta)]\frac{1}{kr}\frac{d}{dr}\\ &\quad \times [rz_n^{(J)}(kr)]\exp(im\phi),\end{aligned} \tag{36}$$

where $\mathbf{i}_\mathbf{r}$, $\mathbf{i}_\theta$, and $\mathbf{i}_\phi$ are unit vectors in the spherical polar coordinate system. Also,

$$\begin{aligned}\pi_{mn}(\cos\theta) &= \frac{m}{\sin\theta}P_n^m(\cos\theta),\\ \tau_{mn}(\cos\theta) &= \frac{d}{d\theta}P_n^m(\cos\theta),\end{aligned} \tag{37}$$

with $0 \le n < \infty$ and $|m| \le n$. The superscript (J) has the same definition as in Eq. (16).

In multisphere scattering theory, the electromagnetic waves scattered by an individual sphere in a cluster need to be transformed into incident waves for other spheres in the cluster. One needs to describe the same scattered field in alternative forms, each referring to a different coordinate system but with exactly the same common basis vector wave functions. The connections between the alternate representations of the same field are provided by the addition theorems, i.e., the expansion of the basis set of one representation in terms of the basis set of another. The vector addition theorem under translation of coordinate system can be represented by [25]

$$\begin{aligned}\mathbf{M}_{\mu\nu}^{(J)l} &= \sum_{n=1}^{\infty}\sum_{m=-n}^{n}[A_{mn}^{\mu\nu}\mathbf{M}_{mn}^{(1)j} + B_{mn}^{\mu\nu}\mathbf{N}_{mn}^{(1)j}], \quad r \le d_{lj},\\ \mathbf{N}_{\mu\nu}^{(J)l} &= \sum_{n=1}^{\infty}\sum_{m=-n}^{n}[B_{mn}^{\mu\nu}\mathbf{M}_{mn}^{(1)j} + A_{mn}^{\mu\nu}\mathbf{N}_{mn}^{(1)j}], \quad r \le d_{lj},\end{aligned} \tag{38}$$

$$\begin{aligned}\mathbf{M}_{\mu\nu}^{(J)l} &= \sum_{n=1}^{\infty}\sum_{m=-n}^{n}[A_{mn}^{\mu\nu}\mathbf{M}_{mn}^{(J)j} + B_{mn}^{\mu\nu}\mathbf{N}_{mn}^{(J)j}], \quad r \ge d_{lj},\\ \mathbf{N}_{\mu\nu}^{(J)l} &= \sum_{n=1}^{\infty}\sum_{m=-n}^{n}[B_{mn}^{\mu\nu}\mathbf{M}_{mn}^{(J)j} + A_{mn}^{\mu\nu}\mathbf{N}_{mn}^{(J)j}], \quad r \ge d_{lj},\end{aligned} \tag{39}$$

where $\mathbf{M}_{\mu\nu}^{(J)l}$ and $\mathbf{N}_{\mu\nu}^{(J)l}$ are the basis vector spherical wave functions about the origin O in the lth coordinate system while $\mathbf{M}_{mn}^{(J)j}$ and $\mathbf{N}_{mn}^{(J)j}$ are about the origin O' in the jth coordinate system; $A_{mn}^{\mu\nu}$ and $B_{mn}^{\mu\nu}$ are the so-called vector translational addition coefficients from the lth to the jth coordinate system. These coefficients depend on the relative separation and direction of the displaced origin O' with respect to O.

B. *Analytical Expressions for the Vector Translational Addition Coefficients*

Three sets of formulations for the vector translational addition coefficients exist in the literature, given by Stein, Cruzan, and Mackowski, respectively. As it will soon become apparent, a careful examination and comparison of these formulations, and a rederivation of Cruzan's expressions are necessary.

1. Stein's and Mackowski's Formulations. Both Stein and Mackowski expressed the vector translational addition coefficients in terms of scalar translational addition coefficients. Stein's expressions are as follows:

$$A_{mn}^{\mu\nu} = C_{mn}^{\mu\nu} + kd_{lj}\cos\theta_{lj} \times \left[\frac{n-m}{n(2n-1)}C_{mn-1}^{\mu\nu} + \frac{n+m+1}{(2n+3)(n+1)}C_{mm+1}^{\mu\nu}\right] + \frac{kd_{lj}}{2}\sin\theta_{lj}\left\{\left[\frac{1}{n(2n-1)}C_{m-1n-1}^{\mu\nu} - \frac{1}{(2n+3)(n+1)}C_{m-1n+1}^{\mu\nu}\right]\exp(-i\phi_{lj}) - \left[\frac{(n-m-1)(n-m)}{n(2n-1)}C_{m+1n-1}^{\mu\nu} - \frac{(n+m+2)(n+m+1)}{(2n+3)(n+1)}C_{m+1n+1}^{\mu\nu}\right]\exp(i\phi_{lj})\right\}, \tag{40}$$

$$B_{mn}^{\mu\nu} = \frac{ikd_{lj}}{2n(n+1)}\{2mC_{mn}^{\mu\nu}\cos\theta_{lj} - [C_{m-1n-1}^{\mu\nu}\exp(-i\phi_{lj}) + (n+m+1)(n-m) \times C_{m+1n-1}^{\mu\nu}\exp(i\phi_{lj})]\sin\theta_{lj}\}. \tag{41}$$

Mackowski's equations are

$$A_{mn}^{\mu\nu} = \frac{(n-m)(n+m+1)C_{m+1n}^{\mu+1\nu} + 2\mu m C_{mn}^{\mu\nu} + (\nu+\mu)(\nu-\mu+1)C_{m-1n}^{\mu-1\nu}}{2n(n+1)}, \tag{42}$$

$$B_{mn}^{\mu\nu} = -\frac{i(2n+1)}{2n(n+1)(2n-1)}[(n-m)(n-m-1)C_{m+1n-1}^{\mu+1\nu} + 2\mu(n-m)C_{mn-1}^{\mu\nu} - (\nu+\mu)(\nu-\mu+1)C_{m-1n-1}^{\mu-1\nu}], \tag{43}$$

or alternatively,

$$B_{mn}^{\mu\nu} = \frac{i(2n+1)}{2n(n+1)(2n+3)}[(n+m+1)(n+m+2)C_{m+1n+1}^{\mu+1\nu} - 2\mu(n+m+1)C_{mn+1}^{\mu\nu} - (\nu+\mu)(\nu-\mu+1)C_{m-1n+1}^{\mu-1\nu}]. \tag{44}$$

Stein and Mackowski expressed the vector translational addition coefficients in terms of seven and six scalar translational addition coefficients, respectively. It is worth noting here that unlike Stein's formulae, the geometrical positions of the spheres do not explicitly appear in Mackowski's equations. This makes Mackowski's formulation more convenient in practical computations. In fact, as Mackowski [21] pointed out, these two formulations can be converted to each other by the use of the equations,

$$C_{m+1n}^{\mu+1\nu} = \frac{ZC_{mn+1}^{\mu\nu} + (n+m+2)\xi C_{m+1n+1}^{\mu\nu}}{2n+3} + \frac{ZC_{mn-1}^{\mu\nu} - (n-m-1)\xi C_{m+1n-1}^{\mu\nu}}{2n-1} + C_{mn}^{\mu\nu}, \tag{45}$$

$$C_{m-1n}^{\mu-1\nu} = \frac{1}{(\nu-\mu+1)(\nu+\mu)}\left\{\frac{n+m}{2n+3}[(n+m+1)ZC_{mn+1}^{\mu\nu} - \eta C_{m-1n+1}^{\mu\nu}] + \frac{n-m+1}{2n-1}[(n-m)ZC_{m-1n}^{\mu\nu} + \eta C_{m-1n-1}^{\mu\nu}] + (n+m)(n-m+1)C_{mn}^{\mu\nu}\right\}, \tag{46}$$

$$C_{mn}^{\mu\nu} = \frac{1}{2(m-\mu)}\left[\frac{\eta C_{m-1n+1}^{\mu\nu} + (n+m+1)(n+m+2)C_{m+1n+1}^{\mu\nu}}{2n+3} + \frac{\eta C_{m-1n-1}^{\mu\nu} + (n-m)(n-m-1)C_{m+1n-1}^{\mu\nu}}{2n-1}\right], \tag{47}$$

where (X, Y, Z) are the Cartesian coordinates of the origin of the jth coordinate system in the lth coordinate system and

$$\xi = X + iY, \quad \eta = X - iY. \tag{48}$$

2. Cruzan's Formulation. Cruzan expressed the vector translational coefficients in terms of the Wigner $3jm$ symbols:

$$A_{mn}^{\mu\nu} = (-1)^m i^{n-\nu}\frac{2n+1}{2n(n+1)}\sum_{p=|n-\nu|}^{n+\nu} i^p[n(n+1) + \nu(\nu+1) - p(p+1)] \times a(-m, n, \mu, \nu, p)\begin{pmatrix} z_p^{(J)}(kd_{lj}) \\ j_p(kd_{lj}) \end{pmatrix}$$

$$\times P_p^{\mu-m}(\cos\theta_{lj})\exp[i(\mu-m)\phi_{lj}] \quad \begin{pmatrix} r\le d_{lj} \\ r\ge d_{lj}\end{pmatrix}, \tag{49}$$

$$B_{mn}^{\mu\nu} = (-1)^{m+1} i^{n-\nu}\frac{2n+1}{2n(n+1)}\sum_{p=|n-\nu|}^{n+\nu} i^p[p^2-(n-\nu)^2]^{1/2}[(n+\nu+1)^2-p^2]^{1/2} \times b(-m,n,\mu,\nu,p,p-1)\begin{pmatrix} z_p^{(J)}(kd_{lj}) \\ j_p(kd_{lj})\end{pmatrix} \times P_p^{\mu-m}(\cos\theta_{lj})\exp[i(\mu-m)\phi_{lj}] \quad \begin{pmatrix} r\le d_{lj} \\ r\ge d_{lj}\end{pmatrix}, \tag{50}$$

where the Gaunt coefficient $a(-m, n, \mu, \nu, p)$ is given by

$$a(-m,n,\mu,\nu,p) = (-1)^{\mu-m}(2p+1)\left[\frac{(n-m)!(\nu+\mu)!(p+m-\mu)!}{(n+m)!(\nu-\mu)!(p-m+\mu)!}\right]^{1/2} \times \begin{pmatrix} n & \nu & p \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} n & \nu & p \\ -m & \mu & m-\mu\end{pmatrix}, \tag{51}$$

and

$$b(-m,n,\mu,\nu,p,p-1) = (-1)^{\mu-m}(2p+1)\left[\frac{(n-m)!(\nu+\mu)!(p+m-\mu)!}{(n+m)!(\nu-\mu)!(p-m+\mu)!}\right]^{1/2} \times \begin{pmatrix} n & \nu & p \\ -m & \mu & m-\mu\end{pmatrix}\begin{pmatrix} n & \nu & p-1 \\ 0 & 0 & 0\end{pmatrix}, \tag{52}$$

or

$$\begin{aligned}[p^2-(n-\nu)^2]^{1/2}&[(n+\nu+1)^2-p^2]^{1/2} b(-m,n,\mu,\nu,p,p-1) \\ &= \frac{2p-1}{2p+1}[(n+m)(n-m+1)a(m-1,n,\mu,\nu,p-1) \\ &\quad -(p-m-\mu)(p-m-\mu-1)a(m+1,n,\mu,\nu,p-1) \\ &\quad -2m(p-m-\mu)a(m,n,\mu,\nu,p-1)].\end{aligned} \tag{53}$$

Examining these expressions, we see that only one equivalent scalar translational addition coefficient appears in each expression. This greatly reduces the computing time requirement, compared to Stein's or Mackowski's formulation. However, Cruzan's expression for $B_{mn}^{\mu\nu}$ is misleading. The Wigner $3jm$ symbol has a symmetry property [35, 36]

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix} = (-1)^{j_1+j_2+j_3}\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3\end{pmatrix}, \tag{54}$$

which implies that

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0\end{pmatrix} \equiv 0, \tag{55}$$

if $j_1 + j_2 + j_3$ is an odd integer. In the expression (52), $p = n + \nu - 2q$, so that $n + \nu + p - 1$ is always odd and, as a consequence,

$$\begin{pmatrix} n & \nu & p-1 \\ 0 & 0 & 0\end{pmatrix} \equiv 0, \tag{56}$$

which leads to $b(-m, n, \mu, \nu, p, p-1) \equiv 0$ and $B_{mn}^{\mu\nu} \equiv 0$. Equation (53) yields the same result. In the right-hand side of Eq. (53), there are three Gaunt coefficients $a(n, \nu, p-1)$, where $p = n + \nu, n + \nu - 2, \ldots$. All these three Gaunt coefficients $a(n, \nu, p-1)$ are identically zero due to the fact that the Gaunt coefficients $a(n, \nu, p)$ vanish whenever $p = n + \nu - 1, n + \nu - 3, \ldots$, as mentioned in Section II. Again, this leads to $B_{mn}^{\mu\nu} \equiv 0$. Obviously, such is not the case. The vector translational addition coefficients $B_{mn}^{\mu\nu}$ are simply not always zero. Cruzan's expressions for $B_{mn}^{\mu\nu}$, Eqs. (50), (52), and (53), should therefore be avoided.

C. *Rederivation of the Expressions for Vector Addition Coefficients*

We have rederived the analytical expressions for both $A_{mn}^{\mu\nu}$ and $B_{mn}^{\mu\nu}$ which are similar to Cruzan's forms.

Stratton [41] showed that the two independent vector solutions of vector wave equations can be written in the form

$$\begin{aligned}\mathbf{M}_{mn}^{(J)} &= \nabla u_{mn}^{(J)} \times \mathbf{r}, \\ \mathbf{N}_{mn}^{(J)} &= \frac{1}{k}\nabla u_{mn}^{(J)} \times \mathbf{M}_{mn}^{(J)},\end{aligned} \tag{57}$$

where $\mathbf{r}$ is the position vector. In the following derivations, we will suppress the superscripts of $\mathbf{M}$, $\mathbf{N}$, and u for brevity. For a point in space, the position vector in the lth coordinate system, $\mathbf{r}_l$, is related to the position vector in the jth coordinate system, $\mathbf{r}_j$, by

$$\mathbf{r}_l = \mathbf{r}_j + \mathbf{r}_{lj}, \tag{58}$$

where $\mathbf{r}_{lj}$ extends from the origin of the lth coordinate system to the origin of the jth coordinate system. Since the gradient of a scalar quantity is invariant to a transformation of the coordinate system, we conclude from the scalar addition theorem (17), the first equation in (57), and Eq. (58) that

$$\mathbf{M}_{\mu\nu}(r_l, \theta_l, \phi_l) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_{mn}^{\mu\nu}[\mathbf{M}_{mn}(r_j, \theta_j, \phi_j) + \nabla u_{mn}(r_j, \theta_j, \phi_j) \times \mathbf{r}_{lj}]. \quad (59)$$

Stein showed that

$$\begin{aligned}\nabla u_{mn}(r_j, \theta_j, \phi_j) \times \mathbf{r}_{lj} &= \frac{kd_{lj}}{2n+1}\Bigg\{\cos\theta_{lj}\left[\frac{n+m}{n}\mathbf{M}_{mn-1} + \frac{n-m+1}{n+1}\mathbf{M}_{mn+1}\right] \\ &+ \frac{\sin\theta_{lj}\exp(-i\phi_{lj})}{2}\left[-\frac{1}{n}\mathbf{M}_{m+1n-1} + \frac{1}{n+1}\mathbf{M}_{m+1n+1}\right] \\ &+ \frac{\sin\theta_{lj}\exp(i\phi_{lj})}{2}\left[\frac{(n+m)(n+m-1)}{n}\mathbf{M}_{m-1n-1}\right. \\ &\left.- \frac{(n-m+1)(n-m+2)}{n+1}\mathbf{M}_{m-1n+1}\right]\Bigg\} \\ &+ \frac{kd_{lj}}{n(n+1)}\Bigg\{m\cos\theta_{lj}\,\mathbf{N}_{mn} - \frac{\sin\theta_{lj}}{2} \\ &\times [\exp(-i\phi_{lj})\mathbf{N}_{m+1n} \\ &+ \exp(i\phi_{lj})(n+m)(n-m+1)\mathbf{N}_{m-1n}]\Bigg\}, \end{aligned} \quad (60)$$

where $\mathbf{M}$ and $\mathbf{N}$ are the abbreviations of $\mathbf{M}(r_j, \theta_j, \phi_j)$ and $\mathbf{N}(r_j, \theta_j, \phi_j)$, respectively. From this Stein obtained the final expressions for the vector translational coefficients shown in Eqs. (40) and (41). His results can be simplified further. By taking the gradient of u_{mn}, resolving it into Cartesian components, and utilizing the recurrence relations for Bessel and Legendre functions, Mackowski has obtained the following three relations:

$$(\mathbf{i_x} + i\mathbf{i_y}) \cdot \nabla u_{mn} = (2n+1)^{-1}(u_{m+1n-1} + u_{m+1n+1}), \quad (61)$$

$$\begin{aligned}(\mathbf{i_x} - i\mathbf{i_y}) \cdot \nabla u_{mn} = (2n+1)^{-1}[&(n+m)(n+m-1)u_{m-1n-1} \\ &+ (n-m+1)(n-m+2)u_{m-1n+1}], \end{aligned} \quad (62)$$

$$\begin{aligned}\mathbf{i_z} \cdot \nabla u_{mn} = (2n+1)^{-1}[&(n+m)u_{m-1n-1} \\ &- (n-m+1)u_{m-1n+1}]. \end{aligned} \quad (63)$$

Additional relations for the vector spherical harmonics are

$$(\mathbf{i_x} + i\mathbf{i_y}) \cdot \mathbf{M}_{mn} = iu_{m+1n}, \quad (64)$$

$$(\mathbf{i_x} - i\mathbf{i_y}) \cdot \mathbf{M}_{mn} = i(n+m)(n-m+1)u_{m-1n}, \quad (65)$$

$$\mathbf{i_z} \cdot \mathbf{M}_{mn} = -imu_{mn}, \quad (66)$$

and

$$(\mathbf{i_x} + i\mathbf{i_y}) \cdot \mathbf{N}_{mn} = (2n+1)^{-1}[nu_{m+1n+1} - (n+1)u_{m+1n-1}], \quad (67)$$

$$\begin{aligned}(\mathbf{i_x} - i\mathbf{i_y}) \cdot \mathbf{N}_{mn} &= (2n+1)^{-1}[n(n-m+1)(n-m+2)u_{m-1n+1} \\ &- (n+1)(n+m)(n+m-1)u_{m-1n-1}], \end{aligned} \quad (68)$$

$$\begin{aligned}\mathbf{i_z} \cdot \mathbf{N}_{mn} = (2n+1)^{-1}[&n(n-m+1)u_{mn+1} \\ &+ (n+1)(n+m)u_{mn-1}]. \end{aligned} \quad (69)$$

Also, four three-term recurrence relations can be obtained [26] from the definition of the Gaunt coefficient, the product derivative

$$\frac{d}{d\theta}(P_n^m P_\nu^\mu) = P_n^m \frac{dP_\nu^\mu}{d\theta} + P_\mu^\nu \frac{dP_n^m}{d\theta} \quad (70)$$

and the derivative relations for the associated Legendre functions

$$\begin{aligned}\frac{dP_n^m}{d\theta} &= \frac{m\cos\theta}{\sin\theta}P_n^m - P_n^{m+1}, \\ \frac{dP_n^m}{d\theta} &= -\frac{m\cos\theta}{\sin\theta}P_n^m + (n+m)(n-m+1)P_n^{m+1}, \end{aligned} \quad (71)$$

namely,

$$\begin{aligned}a(m, n, \mu, \nu, p) = a(&m+1, n, \mu, \nu, p) \\ &+ a(m, n, \mu+1, \nu, p), \end{aligned} \quad (72)$$

$$\begin{aligned}(p-m-\mu+1)&(p+m+\mu)a(m, n, \mu, \nu, p) \\ &= (\nu+\mu)(\nu-\mu+1)a(m, n, \mu-1, \nu, p) \\ &\quad + (n+m)(n-m+1)a(m-1, n, \mu, \nu, p), \end{aligned} \quad (73)$$

$$\begin{aligned}[(p+m+\mu)&(p-m-\mu+1) + (\nu-\mu)(\nu+\mu+1) \\ &- (n+m)(n-m+1)]\, a(m, n, \mu, \nu, p) \\ &= (\nu+\mu)(\nu-\mu+1)a(m, n, \mu-1, \nu, p) + (p-m-\mu) \\ &\quad \times (p+m+\mu+1)a(m, n, \mu+1, \nu, p), \end{aligned} \quad (74)$$

$$\begin{aligned}[(p+m+\mu)&(p-m-\mu+1) + (n-m)(n+m+1) \\ &- (\nu+\mu)(\nu-\mu+1)]\, a(m, n, \mu, \nu, p) \\ &= (n+m)(n-m+1)a(m-1, n, \mu, \nu, p) + (p-m-\mu) \\ &\quad \times (p+m+\mu+1)a(m+1, n, \mu, \nu, p). \end{aligned} \quad (75)$$

By the use of Eqs. (60)–(69) and (72)–(75) it can be shown that

$$A_{mn}^{\mu\nu} = (-1)^{m+n}a_0\frac{2n+1}{2n(n+1)}\exp[i(\mu-m)\phi_{lj}]\sum_{q=0}^{q_{\max}}(-1)^q\tilde{a}_{1q}$$

$$\times [n(n+1)+\nu(\nu+1)-p(p+1)] \begin{pmatrix} z_p^{(J)}(kd_{lj}) \\ j_p(kd_{lj}) \end{pmatrix}$$

$$\times P_p^{\mu-m}(\cos\theta_{lj}) \quad \begin{pmatrix} r \le d_{lj} \\ r \ge d_{lj} \end{pmatrix}, \tag{76}$$

$$B_{mn}^{\mu\nu} = (-1)^{m+n} i a_0 b_0 \frac{2n+1}{2n(n+1)} \exp[i(\mu-m)\phi_{lj}] \sum_{q=0}^{Q_{\max}} (-1)^q$$

$$\times \{2(n+1)(\nu-\mu)\tilde{a}_{2q}$$

$$-[p(p+3)-\nu(\nu+1)-n(n+3)-2\mu(n+1)]\tilde{a}_{3q}\}$$

$$\times \begin{pmatrix} z_{p+1}^{(J)}(kd_{lj}) \\ j_{p+1}(kd_{lj}) \end{pmatrix} P_{p+1}^{\mu-m}(\cos\theta_{lj}) \quad \begin{pmatrix} r \le d_{lj} \\ r \ge d_{lj} \end{pmatrix}, \tag{77}$$

where

$$a_0 = a(-m, n, \mu, \nu, n+\nu) = \frac{(n+1)_n(\nu+1)_\nu(n+\nu+m-\mu)!}{(n+\nu+1)_{n+\nu}(n+m)!(\nu-\mu)!}, \tag{78}$$

$$b_0 = \frac{(2n+1)(n+\nu+m-\mu+1)}{(2n+2\nu+1)(n+m+1)},$$

$$p = n+\nu-2q,$$

$$q_{\max} = \min\left(n, \nu, \frac{n+\nu-|m-\mu|}{2}\right), \tag{79}$$

$$Q_{\max} = \min\left(n+1, \nu, \frac{n+\nu+1-|m-\mu|}{2}\right),$$

and $\tilde{a}_{1q}$, $\tilde{a}_{2q}$, $\tilde{a}_{3q}$, are the normalized Gaunt coefficients defined by

$$\begin{aligned} \tilde{a}_{1q} &= a(-m, n, \mu, \nu, n+\nu-2q)/a(-m, n, \mu, \nu, n+\nu), \\ \tilde{a}_{2q} &= a(-m-1, n+1, \mu+1, \nu, n+\nu+1-2q)/ \\ &\quad a(-m-1, n+1, \mu+1, \nu, n+\nu+1), \\ \tilde{a}_{3q} &= a(-m, n+1, \mu, \nu, n+\nu+1-2q)/ \\ &\quad a(-m, n+1, \mu, \nu, n+\nu+1). \end{aligned} \tag{80}$$

Equation (76) is exactly the same as Eq. (49), but Eq. (77) must replace Eq. (50) that is Cruzan's defective expression for $B_{mn}^{\mu\nu}$.

V. EVALUATION OF THE VECTOR TRANSLATIONAL ADDITION COEFFICIENTS

As discussed above, both Stein and Mackowski express the vector translational addition coefficients in terms of the scalar translational addition coefficients. Thus, the evaluation of the vector translational addition coefficients by the use of Stein's or Mackowski's formulation is based on the evaluation of the relevant scalar addition coefficients, which can be accomplished by either the method of direct calculation or the recursive approach described in Section III. However, the evaluation of these coefficients by the use of the expressions of Cruzan's type can be carried out only by the method of direct calculation because both the expressions for $A_{mn}^{\mu\nu}$ and $B_{mn}^{\mu\nu}$ are not directly related to the scalar addition coefficients due to some ''additional'' factors, like $[n(n+1)+\nu(\nu+1)-p(p+1)]$ in the expression for $A_{mn}^{\mu\nu}$, under the summation sign. This type of formulation takes a summation over q for a range from 0 to $q_{\max}$ concerning $q_{\max}+1$ Gaunt coefficients that need direct evaluations.

In the practical application for solving multisphere scattering problems, the expressions for vector addition coefficients (40)–(44) and (76)–(77) need to introduce a common factor

$$E_{mn}^{\mu\nu} = i^{\nu-n} \frac{(2\nu+1)(n+m)!(\nu-\mu)!}{(2n+1)(n-m)!(\nu+\mu)!}, \tag{81}$$

the discussion of which can be found in Xu [23]. In all of the following calculations, therefore, we replace $A_{mn}^{\mu\nu}$ and $B_{mn}^{\mu\nu}$ by $E_{mn}^{\mu\nu}A_{mn}^{\mu\nu}$ and $E_{mn}^{\mu\nu}B_{mn}^{\mu\nu}$, respectively. For example, Eqs. (76) and (77) are modified as

$$A_{mn}^{\mu\nu} = \frac{(-1)^m i^{\nu+n}(n+2)_{n-1}(\nu+2)_{\nu+1}(n+\nu+m-\mu)!}{4n(n+\nu+1)_{n+\nu}(n-m)!(\nu+\mu)!}$$

$$\times \exp[i(\mu-m)\phi_{lj}] \sum_{q=0}^{q_{\max}} (-1)^q [n(n+1)+\nu(\nu+1)$$

$$-p(p+1)]\tilde{a}_{1q} \begin{pmatrix} z_p^{(J)}(kd_{lj}) \\ j_p(kd_{lj}) \end{pmatrix} P_p^{\mu-m}(\cos\theta_{lj}) \quad \begin{pmatrix} r \le d_{lj} \\ r \ge d_{lj} \end{pmatrix}, \tag{82}$$

$$B_{mn}^{\mu\nu} = \frac{(-1)^m i^{\nu+n+1}(n+2)_{n+1}(\nu+2)_{\nu+1}(n+\nu+m-\mu+1)!}{4n(n+1)(n+m+1)(n+\nu+2)_{n+\nu+1}(n-m)!(\nu+\mu)!}$$

$$\times \exp[i(\mu-m)\phi_{lj}] \sum_{q=0}^{Q_{\max}} (-1)^q \{2(n+1)(\nu-\mu)\tilde{a}_{2q}$$

$$-[p(p+3)-\nu(\nu+1)-n(n+3)-2\mu(n+1)]\tilde{a}_{3q}\}$$

$$\times \begin{pmatrix} z_{p+1}^{(J)}(kd_{lj}) \\ j_{p+1}(kd_{lj}) \end{pmatrix} P_{p+1}^{\mu-m}(\cos\theta_{lj}) \quad \begin{pmatrix} r \le d_{lj} \\ r \ge d_{lj} \end{pmatrix}, \tag{83}$$

or alternatively, $(d_{lj}, \theta_{lj}, \phi_{lj})$ in both Eqs. (76) and (77) can be replaced by $(d_{jl}, \theta_{jl}, \phi_{jl})$ with $i^{\nu+n}$ in Eq. (76) and $i^{\nu+n+1}$ in Eq. (77) replaced by $(-i)^{\nu+n}$ and $(-i)^{\nu+n+1}$, respectively.

A. *Timing Tests*

We carried out extensive timing tests on an IBM RS6000-340 Workstation. The three sets of expressions for the vector translational coefficients and the two methods for calculating the scalar translational coefficients can be combined into five schemes to evaluate the vector translational coefficients. Table I shows a comparison of the required computing times (in seconds) by each of the five possible schemes. In all the test calculations, we used the Hankel function of the first kind, $h_n^{(1)}$, for $z_n^{(J)}$. In Table I, the first column shows the highest degree $n_{\max}(\nu_{\max})$ reached in the computation; the second column, $N = [n_{\max}(n_{\max} + 2)]^2$, is the total number of vector translational coefficients computed, which is the number of all the possible combinations for the integer group (m, n, μ, ν) from the lowest degree $n = \nu = 1$ to the highest degree $n_{\max} = \nu_{\max}$; each of the next five columns indicates the CPU time spent by the corresponding scheme. The heading of each of these five columns indicates what scheme is used. CX stands for the Cruzan-type formulation that is rederived in this paper, M refers to Mackowski's formulation, and S refers to Stein's. The subscript in the heading indicates whether the method of direct calculation (X) or the recursive approach (M) is used to calculate the scalar translational addition coefficients. In all the test computations for which results are shown in Tables I, II, and III, the origin of the second coordinate system has the spherical coordinates of (2.0, 0.5, 0.5) in the first coordinate system, where the separation is expressed by a dimensionless quantity $kd = 2\pi d/\lambda$ and the two angular coordinates are in units of radian. Table I shows that the computing time requirement by the first scheme is a fraction of all the others.

TABLE I

Timing Test for the Calculation of Vector Translational Addition Coefficients

		CPU (in seconds) on IBM RS6000-340 Workstation				
$n_{\max}$, $\nu_{\max}$	$N_{\mathrm{cof.}}$	CX[a]	M_x[b]	M_M[c]	S_X[d]	S_M[e]
5	1225	1	3	4	4	6
10	14400	22	67	86	97	123
15	65025	157	461	628	689	906
20	193600	685	1940	2755	3012	3994
30	921600	5656	15885	24004	25341	35350

[a] The scheme used the revised version of Cruzan's expressions for vector translational addition coefficients and Xu's algorithm for calculating the Gaunt coefficients.

[b] The scheme used Mackowski's expressions for vector translational addition coefficients and Xu's method of direct calculation for scalar translational addition coefficients.

[c] The scheme used Mackowski's expressions for vector translational addition coefficients and Mackowski's recursive approach to calculating scalar translational addition coefficients.

[d] The scheme used Stein's expressions for vector translational addition coefficients and Xu's method of direct calculation for scalar translational addition coefficients.

[e] The scheme used Stein's expressions for vector translational addition coefficients and Mackowski's recursive approach to calculating scalar translational coefficients.

B. *Numerical Results*

The actual computations show that the numerical results obtained from all five schemes mentioned in the above subsection are usually consistent. Table II lists some examples for the numerical values of $A_{mn}^{\mu\nu}$ and $B_{mn}^{\mu\nu}$. For those and other similar groups of (m, n, μ, ν) listed in Table II, all five schemes provide nearly identical numerical results (agreeing exactly up to at least 10 digits in double precision calculations) but it is noteworthy that for all the numerical values shown in Table II, the magnitudes of $A_{mn}^{\mu\nu}$ and $B_{mn}^{\mu\nu}$ and the magnitudes of the real and imaginary parts of $A_{mn}^{\mu\nu}$ and $B_{mn}^{\mu\nu}$ are not significantly different. There is a particular case ($m = -\mu$ and $n = \nu$) for which the difference between the magnitudes of $A_{mn}^{\mu\nu}$ and of $B_{mn}^{\mu\nu}$, as well as between their real and imaginary parts, becomes large, on the level of greater than 10^{10}. In these particular cases, discrepancies in the numerical values obtained by different schemes for the quantities of smaller magnitudes are obvious. Some examples are given in Table III. These discrepancies are due to roundoff errors only and are not due to errors in the formulations. The three sets of expressions we discussed can be readily converted to each other. These discrepancies are probably unavoidable and may indicate latent numerical problems with large size parameters of the spheres.

VI. EXAMPLES OF PRACTICAL APPLICATIONS

The formulation described in this paper has been used in practical multisphere scattering calculations. Some numerical results in the practical application are presented for illustration. We compared the theoretical predictions by the multisphere scattering theory [23] based on the technique for calculation of the addition coefficients described in this paper with the laboratory scattering measurements obtained by Wang [42] through a microwave analog technique. The practical examples shown here are the angular distributions of the polarization components of scattering intensities i_{11} and i_{22} at a fixed orientation for six chains of spheres, each consisting of 2, 3, or 5 identical spheres in various intersphere separations. These quantities correspond to the scattered field components, perpen-

TABLE II

Examples for Numerical Values of $A_{mn}^{\mu\nu}$ and $B_{mn}^{\mu\nu}$

m	n	μ	ν	$A_{mn}^{\mu\nu}$ Real	$A_{mn}^{\mu\nu}$ Imag.	$B_{mn}^{\mu\nu}$ Real	$B_{mn}^{\mu\nu}$ Imag.
1	2	1	2	.40952764E + 00	−.37687246E − 01	−.95439506E + 00	.14571311E + 00
1	5	2	6	.42246528E + 06	.23079387E + 06	.24466070E + 05	−.44784830E + 05
−2	10	7	15	.13425435E + 14	.62258200E + 14	.20835862E + 13	−.44930709E + 12
10	20	3	22	.12643610E + 58	.33753591E + 58	.18943589E + 57	−.70959965E + 56
−28	30	29	32	−.13928339E − 10	.60669708E − 10	−.31805959E − 13	−.73019004E − 14
22	40	−39	45	.34997554E + 216	.45574651E + 216	−.23949371E + 214	.18391132E + 214
5	50	−22	61	.89567769E + 226	−.12101307E + 227	.50442227E + 224	.37334790E + 224

dicular (i_{11}) or parallel (i_{22}) to the scattering plane that is defined by the x-z-plane. For all cases, the incident plane wave vector is along the z-axis and the spheres are aligned along the x-axis indicated by (h) in the figures. The physical and geometrical parameters of the chains of spheres are listed in Table IV. Figure 1 shows the comparison of our theoretical calculations with the corresponding experimental data for these chains of spheres.

VII. CLOSING REMARKS

We have discussed the formulations and computational methods for the vector translational addition coefficients that are essential to the electromagnetic multisphere-scattering theory and tested the numerical results obtained by different formulations. The combination of the revised expressions (82) and (83) with the direct method for calculating the Gaunt coefficients is the most efficient way to calculate the vector translational coefficients. In actual multisphere scattering calculations, this new formulation can be combined with the rotate-translation-rotate scheme introduced by Mackowski [21] to reduce the actual computing time even further. There is a special case of two spheres restricted to lie along a common z-axis, which has been thoroughly studied by Bruning [32]. This specialization results in a great simplification of the addition theorem.

TABLE III

Examples for Numerical Values of $A_{mn}^{\mu\nu}$ and $B_{mn}^{\mu\nu}$ when $m = -\mu$ and $n = \nu$

$kd = 2.0$, $\theta = \phi = 0.5$ rad		$A_{mn}^{\mu\nu}$ Real	$A_{mn}^{\mu\nu}$ Imag.	$B_{mn}^{\mu\nu}$ Real	$B_{mn}^{\mu\nu}$ Imag.
$m = \mu = 0$	CX	.29696820E + 00	−.19286014E + 18	−.16872592E + 02	−.38363062E − 10
$n = \nu = 10$	M_X	.29696820E + 00	−.19286014E + 18	−.62850909E + 03	.28794983E − 01
	M_M	.29696820E + 00	−.19286014E + 18	.49311307E + 06	.14109514E − 09
	S_X	−.21531319E + 01	−.19286014E + 18	−.21463273E + 04	−.17827818E + 00
	S_M	.20093328E + 02	−.19286014E + 18	.27198735E + 03	−.11795571E − 15
$m = -\mu = -5$	CX	−.40406257E + 34	−.11952693E + 34	.14369891E + 17	−.48577633E + 17
$n = \nu = 20$	M_X	−.40406257E + 34	−.11952693E + 34	.37730561E + 22	−.12769437E + 23
	M_M	−.40406257E + 34	−.11952693E + 34	.11790800E + 20	.23581601E + 20
	S_X	−.40406257E + 34	−.11952693E + 34	−.22703132E + 19	.76804213E + 19
	S_M	−.40406257E + 34	−.11952693E + 34	.30190335E + 17	.24152268E + 20
$m = -\mu = 10$	CX	.18077051E + 110	.27881159E + 110	−.13181202E + 94	.855461747E + 93
$n = \nu = 30$	M_X	.18077051E + 110	.27881159E + 110	−.23907708E + 98	.15540010E + 98
	M_M	.18077051E + 110	.27881159E + 110	.47815415E + 96	.11953854E + 96
	S_X	.18077051E + 110	.27881159E + 110	.94172768E + 97	−.61060187E + 97
	S_M	.18077051E + 110	.27881159E + 110	.72319284E + 97	−.46898304E + 97
$m = -\mu = -35$	CX	−.65142622E − 05	.13748543E − 04	.20997296E − 24	.99488279E − 25
$n = \nu = 40$	M_X	−.65142622E − 05	.13748543E − 04	−.77260453E − 17	−.38630226E − 17
	M_M	−.65142622E − 05	.13748543E − 04	.0000000E + 18	−.13320768E − 18
	S_X	−.65142622E − 05	.13748543E − 04	.26119667E − 18	.12373351E − 18
	S_M	−.65142622E − 05	.13748543E − 04	.39989284E − 21	.33324403E − 22

TABLE IV

The Sphere-System Parameters

ID No.	No. of spheres in the chain	Single sphere size parameter x	Complex refractive index m'-im''	Dimensionless[a] separation parameter kd
1	2	3.083	1.61 $-$ i0.004	6.166
2	2	3.083	1.61 $-$ i0.004	8.031
3	2	4.346	1.63 $-$ i0.010	9.941
4	2	4.346	1.63 $-$ i0.010	10.763
5	3	3.083	1.61 $-$ i0.004	7.525
6	5	2.176	1.629 $-$ i.0125	4.352

[a] d is the center to center separation distance between two neighboring spheres.

In this case, the addition theorem becomes orthogonal in order m, that is, $A_{mn}^{\mu\nu} = B_{mn}^{\mu\nu} \equiv 0$ whenever $m \neq \mu$. In general, the spheres in a cluster will not be aligned along the z-axis. However, one can take advantage of this great reduction of computing time through a rotation of coordinates. The rotational transformation for scalar and vector spherical harmonics is identical and is much simpler than the translational transformation.

We see from the numerical examples (Tables II and III) that there is a huge difference in the magnitudes of the

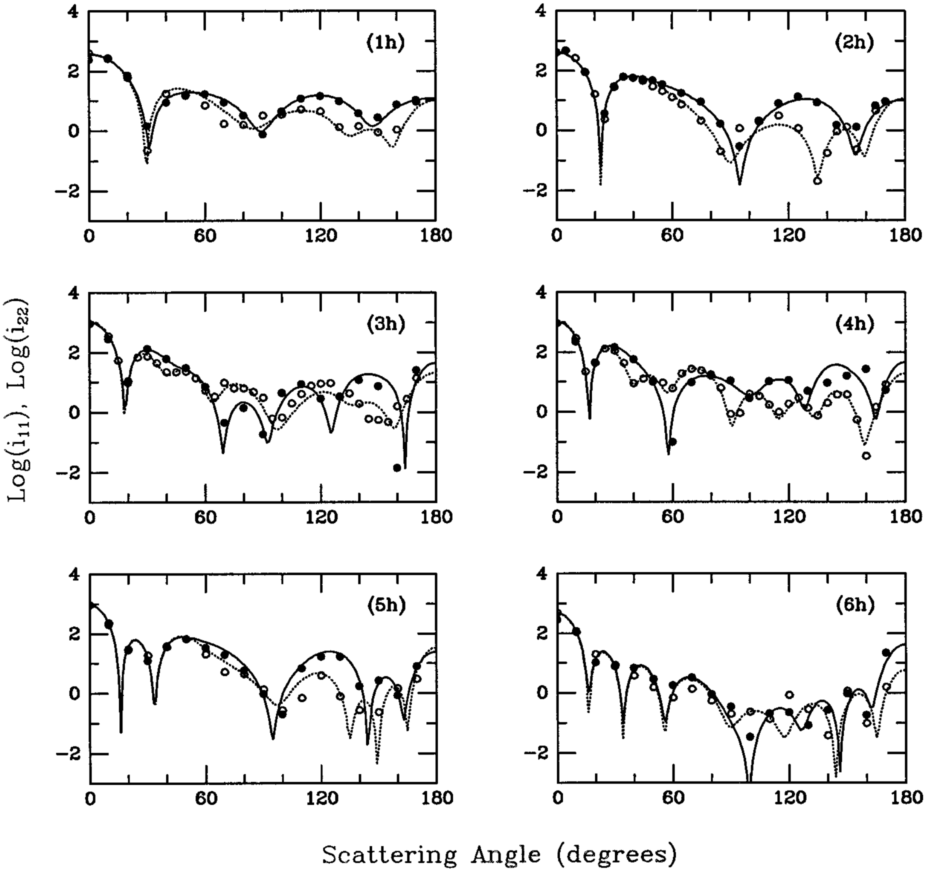

FIG. 1. Angular distributions of six chains of spheres when the axes of symmetry of the chains are aligned along the x-axis (the incident plane wave vector is along the z-axis and the scattering plane is the x-z-plane). The parameters of these sphere chains are listed in Table IV. The number labeled at the upper right corner of each figure is the identification number of the sphere chain involved. A dotted curve in each figure is the theoretical prediction for i_{11} and the continuous curve for i_{22}. The open circles in each figure are the laboratory scattering measurements for i_{11} and the filled circles for i_{22}.

numerical values of vector addition coefficients for low and high degrees. This may be the intrinsic weakness of this multipole expansion method and will impose some limitation on the size parameters of the spheres that the theory can handle.

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